Notes for the course REAL ANALYSIS

5 The spaces L^p

5.1 Basics

Lemma 5.1. *Let $a, b \geq 0$ and $p \geq 1$. Then,*

$$\left(\frac{a+b}{2}\right)^p \leq \frac{a^p + b^p}{2}.$$

Let $a, b \geq 0$ and $p > 1$. Set q such that $1/p + 1/q = 1$. Then,

$$a^{1/p} b^{1/q} \leq \frac{a}{p} + \frac{b}{q}.$$

Proof. **Exercise.** □

Definition 5.2. Let X be a measure space with measure μ and $p > 0$.

$$\mathcal{L}^p(X, \mu, \mathbb{K}) := \{f : X \to \mathbb{K} \,\text{measurable} : |f|^p \,\text{integrable}\}.$$

Define also the function $\|\cdot\|_p : \mathcal{L}^p(X, \mu, \mathbb{K}) \to \mathbb{R}_0^+$ given by

$$\|f\|_p := \left(\int_X |f|^p\right)^{1/p}.$$

Proposition 5.3. *The set $\mathcal{L}^p(X, \mu, \mathbb{K})$ for $p \in (0, \infty)$ is a vector space. Also, $\|\cdot\|_p$ is multiplicative, i.e., $\|\lambda f\|_p = |\lambda| \|f\|_p$ for all $\lambda \in \mathbb{K}$ and $f \in \mathcal{L}^p$. Furthermore, if $p \leq 1$ the function $d_p : \mathcal{L}^p(X, \mu, \mathbb{K}) \times \mathcal{L}^p(X, \mu, \mathbb{K}) \to [0, \infty)$ given by $d_p(f, g) := \|f - g\|_p^p$ is a pseudo-metric (i.e., satisfies the axioms of a metric except for definiteness).*

Proof. **Exercise.** □

Definition 5.4. Let X be a measure space with measure μ. We call a measurable function $f : X \to \mathbb{K}$ *essentially bounded* iff there exists a bounded measurable function $g : X \to \mathbb{K}$ such that $g = f$ almost everywhere. We denote the set of essentially bounded functions by $\mathcal{L}^\infty(X, \mu, \mathbb{K})$. Define also the function $\|\cdot\|_\infty : \mathcal{L}^\infty(X, \mu, \mathbb{K}) \to \mathbb{R}_0^+$ given by

$$\|f\|_\infty := \inf\{\|g\|_{\sup} : g = f \,\text{a.e. and}\, g \,\text{bounded measurable}\}.$$

Proposition 5.5. *The set $\mathcal{L}^\infty(X, \mu, \mathbb{K})$ is a vector space and $\|\cdot\|_\infty$ is a seminorm.*

Proof. **Exercise.** □

Proposition 5.6. *Let f, g be measurable maps such that $f = g$ almost everywhere. Let $p \in (0, \infty]$. Then, $f \in \mathcal{L}^p$ iff $g \in \mathcal{L}^p$.*

Proof. Apply Proposition 4.12 to $|f|^p$ and $|g|^p$. □

Proposition 5.7. *Let $f \in \mathcal{L}^p$ for $p \in (0, \infty)$. Then, f vanishes outside of a σ-finite set.*

Proof. By Proposition 4.13, $|f|^p$ vanishes outside a σ-finite set and hence so does f. □

Proposition 5.8. *Let $f \in \mathcal{L}^\infty$. Then, the set $\{x : |f(x)| > \|f\|_\infty\}$ has measure zero. Moreover, there exists $g \in \mathcal{L}^\infty$ bounded such that $g = f$ almost everywhere and $\|g\|_{sup} = \|g\|_\infty = \|f\|_\infty$.*

Proof. Fix $c > 0$ and consider the set $A_c := \{x : |f(x)| \geq \|f\|_\infty + c\}$. Since there exists a bounded measurable function g such that $g = f$ almost everywhere and $\|g\|_{\text{sup}} < \|f\|_\infty + c$ we must have $\mu(A_c) = 0$. Thus $\{A_{1/n}\}_{n \in \mathbb{N}}$ is an increasing sequence of sets of measure zero. So, their union $A := \bigcup_{n \in \mathbb{N}} A_n = \{x : |f(x)| > \|f\|_\infty\}$ must have measure zero. Define now

$$g(x) := \begin{cases} f(x) & \text{if } x \in X \setminus A \\ 0 & \text{if } x \in A \end{cases}.$$

Then, g is measurable, bounded, and $g = f$ almost everywhere. Moreover, $\|g\|_{\text{sup}} \leq \|f\|_\infty$. On, the other hand, since $g = f$ almost everywhere we must have $\|g\|_{\text{sup}} \geq \|f\|_\infty$ by the definition of $\|\cdot\|_\infty$. Also, $f - g = 0$ almost everywhere and hence $\|f - g\|_\infty \leq \|0\|_{\text{sup}}$, i.e., $\|f - g\|_\infty = 0$ and thus $\|f\|_\infty = \|g\|_\infty$. □

Proposition 5.9. *Let $f \in \mathcal{L}^p$ for $p \in (0, \infty]$. Then $\|f\|_p = 0$ iff $f = 0$ almost everywhere.*

Proof. If $p < \infty$ apply Proposition 4.21 to $|f|^p$. **Exercise.**Complete the proof for $p = \infty$. □

Theorem 5.10 (Hölder's inequality)**.** *Let $p \in [1, \infty]$ and q such that $1/p + 1/q = 1$. Given $f \in \mathcal{L}^p$ and $g \in \mathcal{L}^q$ we have $fg \in \mathcal{L}^1$ and,*

$$\|fg\|_1 \leq \|f\|_p \|g\|_q.$$

Proof. First observe that fg is measurable by Proposition 3.18 since f and g are measurable.

We start with the case $p = 1$ and $q = \infty$. (The case $q = 1$ and $p = \infty$ is analogous.) By Proposition 5.8 there is a bounded function $h \in \mathcal{L}^\infty$ such that $h = g$ almost everywhere and $\|h\|_{\text{sup}} = \|g\|_\infty$. We have

$$|fh| \leq |f| \|h\|_{\text{sup}}.$$

Thus, $|fh|$ is bounded from above by an integrable function and hence fh is integrable by Proposition 4.27. But $fh = fg$ almost everywhere and so fg is integrable by Proposition 4.12. Moreover, integrating the above inequality over X we obtain,

$$\|fg\|_1 = \int_X |fg| = \int_X |fh| \le \|h\|_{\sup} \int_X |f| = \|f\|_1 \|g\|_\infty.$$

It remains to consider the case $p \in (1, \infty)$. If $\|f\|_p = 0$ or $\|g\|_q = 0$ then f or g vansihes almost everywhere by Proposition 5.9. Thus, fg vanishes almost everywhere and $\|fg\|_1 = 0$ by the same Proposition (and in particular $fg \in \mathcal{L}^1$). We thus assume now $\|f\|_p \neq 0$ and $\|g\|_q \neq 0$. Set

$$a := \frac{|f|^p}{\|f\|_p^p}, \quad \text{and} \quad b := \frac{|g|^q}{\|g\|_q^q}.$$

Using the second inequality of Lemma 5.1 we find,

$$\frac{|fg|}{\|f\|_p \|g\|_q} \le \frac{1}{p} \frac{|f|^p}{\|f\|_p^p} + \frac{1}{q} \frac{|g|^q}{\|g\|_q^q}.$$

This implies that $|fg|$ is bounded from above by an integrable function and is hence integrable by Proposition 4.27. Moreover, integrating both sides of the inequality over X yields the inequality that is to be demonstrated. □

Proposition 5.11 (Minkowski's inequality)**.** *Let $p \in [1, \infty]$ and $f, g \in \mathcal{L}^p$. Then,*

$$\|f + g\|_p \le \|f\|_p + \|g\|_p.$$

In particular, $\|\cdot\|_p$ is a seminorm.

Proof. The case $p = 1$ is already implied by Proposition 4.15 while the case $p = \infty$ is implied by Proposition 5.5. We may thus assume $p \in (1, \infty)$. Set q such that $1/p + 1/q = 1$. We have,

$$|f + g|^p \le |f||f + g|^{p-1} + |g||f + g|^{p-1}.$$

Notice that $|f + g|^{p-1} \in \mathcal{L}^q$ so that the two summands on the right hand side are integrable by Theorem 5.10. Integrating on both sides and applying Hölder's inequality to both summands on the right hand side yields,

$$\|f + g\|_p^p \le \|f\|_p \||f + g|^{p-1}\|_q + \|g\|_p \||f + g|^{p-1}\|_q$$

Noticing that $\||f + g|^{p-1}\|_q = \|f + g\|_p^{p-1}$ we find,

$$\|f + g\|_p^p \le (\|f\|_p + \|g\|_p)\|f + g\|_p^{p-1}.$$

Dividing by $\|f + g\|_p^{p-1}$ yields the desired inequality. This is nothing but the triangle inequality for $\|\cdot\|_p$. The other properties making this into a seminorm are immediately verified. □

Theorem 5.12. *Let $p \in [1, \infty)$ and $\{f_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{L}^p$. Then, the sequence converges to some $f \in \mathcal{L}^p$ in the $\|\cdot\|_p$-seminorm. That is, $\mathcal{L}^p$ is complete. Furthermore, there exists a subsequence which converges pointwise almost everywhere to f and for any $\epsilon > 0$ converges uniformly to f outside of a set of measure less than ϵ.*

Proof. Since $\{f_n\}_{n\in\mathbb{N}}$ is Cauchy, there exists a subsequence $\{f_{n_k}\}_{k\in\mathbb{N}}$ such that

$$\|f_{n_l} - f_{n_k}\|_p < 2^{-2k} \quad \forall k \in \mathbb{N} \quad \text{and} \quad \forall l \geq k.$$

Define

$$Y_k := \{x \in X : |f_{n_{k+1}}(x) - f_{n_k}(x)| \geq 2^{-k}\} \quad \forall k \in \mathbb{N}.$$

Then,

$$2^{-kp}\mu(Y_k) \leq \int_{Y_k} |f_{n_{k+1}} - f_{n_k}|^p \leq \int_X |f_{n_{k+1}} - f_{n_k}|^p < 2^{-2kp} \quad \forall k \in \mathbb{N}.$$

This implies, $\mu(Y_k) < 2^{-kp} \leq 2^{-k}$ for all $k \in \mathbb{N}$. Define now $Z_j := \bigcup_{k=j}^{\infty} Y_k$ for all $j \in \mathbb{N}$. Then, $\mu(Z_j) \leq 2^{1-j}$ for all $j \in \mathbb{N}$.

Fix $\epsilon > 0$ and choose $j \in \mathbb{N}$ such that $2^{1-j} < \epsilon$. Let $x \in X \setminus Z_j$. Then, for $k \geq j$ we have

$$|f_{n_{k+1}}(x) - f_{n_k}(x)| < 2^{-k}.$$

Thus, the sum $\sum_{k=1}^{\infty} f_{n_{k+1}}(x) - f_{n_k}(x)$ converges absolutely. In particular, the limit

$$f(x) := \lim_{l\to\infty} f_{n_l}(x) = f_{n_1}(x) + \sum_{l=1}^{\infty} f_{n_{l+1}}(x) - f_{n_l}(x)$$

exists. For all $k \geq j$ we have the estimate,

$$|f(x) - f_{n_k}(x)| = \left|\sum_{l=k}^{\infty} f_{n_{l+1}}(x) - f_{n_l}(x)\right| \leq \sum_{l=k}^{\infty} \left|f_{n_{l+1}}(x) - f_{n_l}(x)\right| \leq 2^{1-k}$$

Thus, $\{f_{n_k}\}_{k\in\mathbb{N}}$ converges to f uniformly outside of Z_j, where $\mu(Z_j) < \epsilon$.

Repeating the argument for arbitrarily small ϵ we find that f is defined on $X \setminus Z$, where $Z := \bigcap_{j=1}^{\infty} Z_j$. Furthermore, $\{f_{n_k}\}_{k\in\mathbb{N}}$ converges to f pointwise on $X \setminus Z$. Note that $\mu(Z) = 0$. By Theorem 3.19, f is measurable on $X \setminus Z$. We extend f to a measurable function on all of X by declaring $f(x) = 0$ if $x \in Z$.

For fixed $k \in \mathbb{N}$ consider the sequence $\{g_l\}_{l\in\mathbb{N}}$ of integrable functions given by

$$g_l := |f_{n_l} - f_{n_k}|^p.$$

Since the sequence $\{\int_X g_l\}_{l\in\mathbb{N}}$ is bounded, $\liminf_{l\to\infty} \int_X g_l$ exists and we can apply Proposition 4.25. Thus, there exists an integrable function g and

$g(x) = \liminf_{l\to\infty} g_l(x)$ almost everywhere. We conclude that $g = |f - f_{n_k}|^p$ almost everywhere. In particular, since g is integrable, $f - f_{n_k} \in \mathcal{L}^p$ and so also $f \in \mathcal{L}^p$. Moreover,

$$\int_X |f - f_{n_k}|^p \le \liminf_{l\to\infty} \int_X |f_{n_l} - f_{n_k}|^p < 2^{-2kp}.$$

In particular,

$$\|f - f_{n_k}\|_p < 2^{-2k}.$$

So $\{f_{n_k}\}_{k\in\mathbb{N}}$ and therefore also $\{f_n\}_{n\in\mathbb{N}}$ converges to f in the $\|\cdot\|_p$-seminorm. □

Theorem 5.13. *Let $\{f_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{L}^\infty$. Then, the sequence converges uniformly almost everywhere to a function $f \in \mathcal{L}^\infty$. Furthermore, the sequence converges to f in the $\mathcal{L}^\infty$-seminorm. In particular, $\mathcal{L}^\infty$ is complete.*

Proof. Define $Z_n := \{x \in X : |f_n(x)| > \|f_n\|_\infty\}$ for all $n \in \mathbb{N}$ and $Y_{n,m} := \{x \in X : |f_n(x) - f_m(x)| > \|f_n - f_m\|_\infty\}$. By Proposition 5.8 $\mu(Z_n) = 0$ for all $n \in \mathbb{N}$ and $\mu(Y_{n,m}) = 0$ for all $n, m \in \mathbb{N}$. Define

$$Z := \left(\bigcup_{n\in\mathbb{N}} Z_n\right) \cup \left(\bigcup_{n,m\in\mathbb{N}} Y_{n,m}\right).$$

Then, $\mu(Z) = 0$. So, $\{f_n(x)\}_{n\in\mathbb{N}}$ converges uniformly on $X \setminus Z$ to some measurable function f. We extend f to a measurable function on all of X by defining $f(x) = 0$ if $x \in Z$. **Exercise.**Complete the proof. □

Theorem 5.14 (Dominated Convergence Theorem in $\mathcal{L}^p$)**.** *Let $p \in [1, \infty)$. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of functions in $\mathcal{L}^p$ such that there exists a real valued function $g \in \mathcal{L}^p$ with $|f_n| \le g$ for all $n \in \mathbb{N}$. Assume also that $\{f_n\}_{n\in\mathbb{N}}$ converges pointwise almost everywhere to a measurable function f. Then, $f \in \mathcal{L}^p$ and $\{f_n\}_{n\in\mathbb{N}}$ converges to f in the $\|\cdot\|_p$-seminorm.*

Proof. **Exercise.**Prove this by suitably adapting the proof of Theorem 4.26. Hint: Replace $|f_n - f_m|$ by $|f_n - f_m|^p$, and apply Theorem 5.12 instead of Proposition 4.22. □

Proposition 5.15. *Let $p \in [1, \infty)$. Then, $\mathcal{S} \subseteq \mathcal{L}^p$ is a dense subset.*

Proof. If f is an integrable simple function f, then $|f|^p$ is also integrable simple. So, $\mathcal{S}$ is a subset of $\mathcal{L}^p$. Now consider $f \in \mathcal{L}^p$. We need to construct a sequence of integrable simple functions that converges to f in the $\|\cdot\|_p$-seminorm. **Exercise.**Do this by appropriately modifying the proof of Proposition 4.27. □

Proposition 5.16. *The simple maps form a dense subset of $\mathcal{L}^\infty$.*

Proof. Let $f \in \mathcal{L}^\infty$ and fix $\epsilon > 0$. The statement follows if we can show that there exists a simple map h such that $\|f-h\|_\infty < \epsilon$. By Proposition 5.8 there is a bounded map $g \in \mathcal{L}^\infty$ such that $g = f$ almost everywhere and $\|g\|_{\sup} = \|f\|_\infty$. Since g is bounded, its image $A \subset \mathbb{K}$ is bounded and thus contained in a compact set. This means that we can cover A by a finite number of open balls $\{B_k\}_{k\in\{1,\ldots,n\}}$ of radius ϵ. Denote the centers of the balls by $\{x_k\}_{k\in\{1,\ldots,n\}}$. Now take measurable subsets $C_k \subseteq B_k$ such that $C_i \cap C_j = \emptyset$ if $i \neq j$ while still covering A, i.e., $A \subseteq \bigcup_{k\in\{1,\ldots,n\}} C_k$. (**Exercise.**Explain how this can be done.) Define $D_k := g^{-1}(C_K)$. $\{D_k\}_{k\in\{1,\ldots,k\}}$ form a measurable partition of X. Now set $h(x) := x_k$ if $x \in D_k$. Then, h is simple and $\|f-h\|_\infty = \|g-h\|_\infty \leq \|g-h\|_{\sup} < \epsilon$. □

Exercise 23. The Monotone Convergence Theorem (Theorem 4.23) and the Dominated Convergence Theorem (Theorem 4.26 or 5.14) are not true in $\mathcal{L}^\infty$. Give a counterexample to both. More precisely, give a pointwise increasing sequence $\{f_n\}_{n\in\mathbb{N}}$ of real non-negative valued functions $f_n \in \mathcal{L}^\infty$ on some measure space X such that $\{f_n\}_{n\in\mathbb{N}}$ converges pointwise to some $f \in \mathcal{L}^\infty$, but $\{f_n\}_{n\in\mathbb{N}}$ does not converge to any function in the $\|\cdot\|_\infty$-seminorm.

5.2 Banach and Hilbert spaces

Definition 5.17. Let V, W be normed vector spaces. Then, a linear map $\alpha : V \to W$ is called *bounded* iff there exists a constant $c \geq 0$ such that

$$\|\alpha(v)\|_W \leq c\|v\|_V \quad \forall v \in V.$$

Proposition 5.18. *Let V, W be normed vector spaces. Then, a linear map $\alpha : V \to W$ is bounded iff it is continuous.*

Proof. **Exercise.** □

A complete normed vector space is also called a *Banach space*. We have seen already that the spaces $\mathcal{L}^p$ with $p \in [1,\infty]$ are vector spaces with a seminorm $\|\cdot\|_p$ and are complete with respect to this seminorm. In order to convert a vector space with a seminorm into a vector space with a norm, we may quotient by those elemnts whose seminorm is zero.

Proposition 5.19. *Let V be a vector space with a seminorm $\|\cdot\|_V$. Consider the subset $A := \{v \in V : \|v\|_V = 0\}$. Then, A is a vector subspace. Moreover $v \sim w \iff v - w \in A$ defines an equivalence relation and $W := V/\sim$ is a vector space. The seminorm $\|\cdot\|_V$ decends to a norm on W via $\|[v]\|_W := \|v\|_V$ for $v \in V$. Also, if V is complete with respect to the seminorm $\|\cdot\|_V$, then W is complete with respect to the norm $\|\cdot\|_W$.*

Proof. If $v \in A$ and $\lambda \in \mathbb{K}$ then $\lambda v \in A$ since $\|\lambda v\|_V = |\lambda|\|v\|_V = 0$. Also, if $v, w \in A$, then $v + w \in A$ because $\|v + w\|_V \leq \|v\|_V + \|w\|_V = 0$ by the triangle inequality. So, A is a vector subspace. That $\sim$ is an equivalence relation follow from the fact that A is a vector space: $\sim$ is refexive because $v - v = 0 \in A$, it is symmetric because from $u - v \in A$ follows $v - u \in A$, and it is transitive because from $u - v \in A$ and $v - w \in A$ follows $u - w \in A$. In order to give W a vector space structure we want to define $\lambda[v] := [\lambda v]$ for $v \in V$ and $\lambda \in \mathbb{K}$ and $[v] + [w] := [v + w]$ for $v, w \in V$. We have to show that these definitions are well. Suppose $v \in V$ and $a \in A$. Then, $\lambda[v + a] = [\lambda v + \lambda a] = [\lambda v] = \lambda[v]$ as required. Similarly, for $v, w \in V$ and $a, b \in A$ we have $[v + a] + [w + b] = [v + w + a + b] = [v + w] = [v] + [w]$ as required. For $\| \cdot \|_W$ we check first that it is well defined. Let $v \in V$ and $a \in A$. Then, $\|[v + a]\|_W = \|v + a\|_V \leq \|v\|_V + \|a\|_V = \|v\|_V = \|[v]\|_W$. But also, $\|[v + a]\|_W = \|v + a\|_V \geq \|v\|_V - \| - a\|_V = \|v\|_V = \|[v]\|_W$. This shows that $\| \cdot \|_W$ is well defined. **Exercise.**Show that $\| \cdot \|_W$ is a norm and that the space W is complete if V is complete. □

Definition 5.20. Let $p \in [1, \infty]$. Then the quotient space $\mathcal{L}^p / \sim$ in the sense of Proposition 5.19 is denoted by L^p. It is a Banach space.

Banach spaces have many useful properties that make it easy to work with them. So usually, one works with the spaces L^p instead of the spaces $\mathcal{L}^p$. Nevertheless one can still think of the these as "spaces of functions" even though they are spaces of equivalence classes. But (because of Proposition 5.9) two functions are in one equivalence class only if they are "essentially the same", i.e., equal almost everywhere.

Proposition 5.21. *Let $p, q \in (0, \infty]$ and set $r \in (0, \infty]$ such that $1/r = 1/p + 1/q$. Then, given $f \in \mathcal{L}^p$ and $g \in \mathcal{L}^q$ we have $fg \in \mathcal{L}^r$. Moreover, the following inequality holds,*

$$\|fg\|_r \leq \|f\|_p \|g\|_q.$$

Proof. **Exercise.**[Hint: For $f \in \mathcal{L}^p$ and $g \in \mathcal{L}^q$ apply Hölder's Theorem (Theorem 5.10) to $|f|^r$ and $|g|^r$, in the case $r < \infty$. Treat the case $r = \infty$ separately.] □

Proposition 5.22. *Let $0 < p \leq q < r \leq \infty$. Then, $\mathcal{L}^p \cap \mathcal{L}^r \subseteq \mathcal{L}^q$. Moreover, if $r < \infty$,*

$$\|f\|_q^{q(r-p)} \leq \|f\|_p^{p(r-q)} \|f\|_r^{r(q-p)} \quad \forall f \in \mathcal{L}^p \cap \mathcal{L}^r.$$

If $r = \infty$ we have,

$$\|f\|_q^q \leq \|f\|_p^p \|f\|_\infty^{q-p} \quad \forall f \in \mathcal{L}^p \cap \mathcal{L}^\infty.$$

If $p \geq 1$, then also $\mathrm{L}^p \cap \mathrm{L}^r \subseteq \mathrm{L}^q$.

Proof. **Exercise**. □

Proposition 5.23. *Let X be a measure space with finite measure μ. Let $0 < p \le q \le \infty$. Then, $\mathcal{L}^q(X,\mu) \subseteq \mathcal{L}^p(X,\mu)$. Moreover,*

$$\|f\|_p \le \|f\|_q \, (\mu(X))^{1/p-1/q} \quad \forall f \in \mathcal{L}^q(X,\mu).$$

If $p \ge 1$, then also $\mathrm{L}^q(X,\mu) \subseteq \mathrm{L}^p(X,\mu)$.

Proof. **Exercise**. □

Definition 5.24. Let V be a complex vector space and $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{C}$ a map. $\langle\cdot,\cdot\rangle$ is called a *sesquilinear* form iff it satisifes the following properties:

- $\langle u+v, w\rangle = \langle u,w\rangle + \langle v,w\rangle$ and
  $\langle u, v+w\rangle = \langle u,v\rangle + \langle u,w\rangle$ for all $u,v,w \in V$.
- $\langle \lambda u, v\rangle = \lambda\langle u,v\rangle$ and $\langle u, \lambda v\rangle = \overline{\lambda}\langle u,v\rangle$ for all $\lambda \in \mathbb{C}$ and $v \in V$.

$\langle\cdot,\cdot\rangle$ is called *hermitian* iff it satisifes in addition the following property:

- $\langle u,v\rangle = \overline{\langle v,u\rangle}$ for all $u,v \in V$.

$\langle\cdot,\cdot\rangle$ is called *positive* iff it satisifes in addition the following property:

- $\langle v,v\rangle \ge 0$ for all $v \in V$.

$\langle\cdot,\cdot\rangle$ is called *definite* iff it satisifes in addition the following property:

- If $\langle v,v\rangle = 0$ then $v = 0$ for all $v \in V$.

Proposition 5.25 (from Lang)**.** *Let V be a complex vector space with a positive hermitian sesquilinear form $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{C}$. If $v \in V$ is such that $\langle v,v\rangle = 0$, then $\langle v,w\rangle = \langle w,v\rangle = 0$ for all $w \in V$.*

Proof. Suppose $\langle v,v\rangle = 0$ for a fixed $v \in V$. Fix some $w \in V$. For any $t \in \mathbb{R}$ we have,

$$0 \le \langle tv+w, tv+w\rangle = 2t\,\Re(\langle v,w\rangle) + \langle w,w\rangle.$$

If $\Re(\langle v,w\rangle) \neq 0$ we could find $t \in \mathbb{R}$ such that the right hand side would be negative, a contradiction. Hence, we can conclude $\Re(\langle v,w\rangle) = 0$, for all $w \in V$. Thus, also $0 = \Re(\langle v, \mathrm{i}w\rangle) = \Re(-\mathrm{i}\langle v,w\rangle) = \Im(\langle v,w\rangle)$ for all $w \in V$. Hence, $\langle v,w\rangle = 0$ and $\langle w,v\rangle = \overline{\langle v,w\rangle} = 0$ for all $w \in V$. □

Theorem 5.26 (Schwarz Inequality)**.** *Let V be a complex vector space with a positive hermitian sesquilinear form $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{C}$. Then, the following inequality is satisfied:*

$$|\langle v,w\rangle|^2 \le \langle v,v\rangle\langle w,w\rangle \quad \forall v,w \in V.$$

Proof. If $\langle v, v\rangle = 0$ then also $\langle v, w\rangle = 0$ by Proposition 5.25 and the inequality holds. Thus, we may assume $\alpha := \langle v, v\rangle \neq 0$ and we set $\beta := -\langle w, v\rangle$. By positivity we have,

$$0 \leq \langle \beta v + \alpha w, \beta v + \alpha w\rangle.$$

Using sesquilinearity and hermiticity on the right hand side this yields,

$$0 \leq |\langle v, v\rangle|^2 \langle w, w\rangle - \langle v, v\rangle |\langle v, w\rangle|^2.$$

(**Exercise.**Show this.) Since $\langle v, v\rangle \neq 0$ we can divide by it and arrive at the required inequality. □

Proposition 5.27. *Let V be a complex vector space with a positive hermitian sesquilinear form $\langle \cdot, \cdot\rangle : V \times V \to \mathbb{C}$. Then, V carries a seminorm given by $\|v\| := \sqrt{\langle v, v\rangle}$. If $\langle \cdot, \cdot\rangle$ is also definite then $\|\cdot\|$ is a norm.*

Proof. **Exercise.**Hint: To prove the triangle inequality, show that $\|v+w\|^2 \leq (\|v\|+\|w\|)^2$ can be derived from the Schwarz inequality (Theorem 5.26). □

Definition 5.28. A positive definite hermitian sesquilinear form is also called an *inner product* or a *scalar product*. A complex vector space equipped with such a form is called an *inner product space* or a *pre-Hilbert space*. It is called a *Hilbert space* iff it is complete with respect to the induced norm.

Theorem 5.29. *Let H be a Hilbert space and $\alpha : H \to \mathbb{K}$ a bounded linear map. Then, there exists a unique element $w \in H$ such that*

$$\alpha(v) = \langle v, w\rangle \quad \forall v \in H.$$

Proposition 5.30. *Consider the map $\langle \cdot, \cdot\rangle : \mathcal{L}^2 \times \mathcal{L}^2 \to \mathbb{C}$ given by*

$$\langle f, g\rangle := \int f\overline{g}.$$

Then, $\langle \cdot, \cdot\rangle$ is a positive hermitian sesquilinear form on $\mathcal{L}^2$. Moreover, the seminorm induced by it according to Proposition 5.27 is the $\|\cdot\|_2$-seminorm. Also, the map $\langle \cdot, \cdot\rangle : \mathrm{L}^2 \times \mathrm{L}^2 \to \mathbb{C}$ given by $\langle [f], [g]\rangle := \langle f, g\rangle$ defines a positive definite hermitian sesquilinear form on L^2. The norm induced by it is the $\|\cdot\|_2$-norm. This makes L^2 into a Hilbert space.

Proof. **Exercise.** □

5.3 Relations between measures

Proposition 5.31. *Let X be a measured space with σ-algebra $\mathcal{M}$. Let μ_1, μ_2 be positive measures on $\mathcal{M}$. Then, $\mu := \mu_1 + \mu_2$ is a positive measure on $(X, \mathcal{M})$. Moreover, $\mathcal{L}^1(\mu) = \mathcal{L}^1(\mu_1) \cap \mathcal{L}^1(\mu_2)$ and*

$$\int_A f \, \mathrm{d}\mu = \int_A f \, \mathrm{d}\mu_1 + \int_A f \, \mathrm{d}\mu_2 \quad \forall f \in \mathcal{L}^1(\mu), A \in \mathcal{M}.$$

Proof. **Exercise.** □

Definition 5.32 (Complex Measure)**.** Let X be a measured space with σ-algebra $\mathcal{M}$. Then, a map $\mu : \mathcal{M} \to \mathbb{C}$ is called a *complex measure* iff it is countably additive, i.e., satisfies the following property: If $\{A_n\}_{n\in\mathbb{N}}$ is a sequence of elements of $\mathcal{M}$ such that $A_n \cap A_m = \emptyset$ if $n \neq m$, then

$$\mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n).$$

Remark 5.33. 1. The above definition implies $\mu(\emptyset) = 0$. 2. The convergence of the series in the definition is absolute since its limit must be invariant under reorderings. 3. In contrast to positive measures, a complex measure is always finite.

Exercise 24. Show that the complex measures on a given σ-algebra form a complex vector space.

Definition 5.34. Let X be a measured space with σ-algebra $\mathcal{M}$. Let μ be a positive measure on $(X, \mathcal{M})$ and ν a positive or complex measure on $(X, \mathcal{M})$. We say that ν is *absolutely continuous* with respect to μ, denoted $\nu \ll \mu$ iff $\mu(A) = 0$ implies $\nu(A) = 0$ for all $A \in \mathcal{M}$.

Definition 5.35. Let X be a measured space with σ-algebra $\mathcal{M}$. Let μ be a positive or complex measure on $(X, \mathcal{M})$. We say that μ is *concentrated* on $A \in \mathcal{M}$ iff $\mu(B) = \mu(B \cap A)$ for all $B \in \mathcal{M}$.

Definition 5.36. Let X be a measured space with σ-algebra $\mathcal{M}$. Let μ, ν be positive or complex measures on $(X, \mathcal{M})$. We say that μ and ν are *mutually singular*, denoted $\mu \perp \nu$, iff there exist disjoint sets $A, B \in \mathcal{M}$ such that μ is concentrated on A and ν is concentrated on B.

Proposition 5.37. *Let μ be a positive measure and ν, ν_1, ν_2 be positive or complex measures.*

1. *If μ is concentrated on A and $\nu \ll \mu$, then ν is concentrated on A.*
2. *If $\nu_1 \ll \mu$ and $\nu_2 \perp \mu$, then $\nu_1 \perp \nu_2$.*
3. *If $\nu \ll \mu$ and $\nu \perp \mu$, then $\nu = 0$.*

4. If $\nu_1 \ll \mu$ and $\nu_2 \ll \mu$, then $\nu_1 + \nu_2 \ll \mu$.

5. If $\nu_1 \perp \nu$ and $\nu_2 \perp \nu$, then $\nu_1 + \nu_2 \perp \nu$.

Proof. **Exercise.** □

Theorem 5.38 (Averaging Theorem)**.** *Let X be a measure space with σ-finite measure μ. Let $S \subseteq \mathbb{K}$ be a closed subset and $f \in \mathcal{L}^1(X, \mu, \mathbb{K})$. If for any measurable set A of finite measure we have*

$$\frac{1}{\mu(A)} \int_A f \mathrm{d}\mu \in S,$$

then $f(x) \in S$ for almost all $x \in X$.

Proof. Let $C := \{x \in X : f(x) \notin S\}$. We need to show that $\mu(C) = 0$. Assume the contrary, i.e., $\mu(C) > 0$. Write $\mathbb{K} \setminus S = \bigcup_{n \in \mathbb{N}} B_n$ as a countable union of open balls $\{B_n\}_{n \in \mathbb{N}}$. Their preimages are measurable and cover C. There is at least one open ball B_n such that $\mu(f^{-1}(B_n)) > 0$. Say this open ball has centre x and radius r. Furthermore, there is a measurable subset $D \subseteq f^{-1}(B_n)$ such that $0 < \mu(D) < \infty$. Then,

$$\begin{aligned}\left|\frac{1}{\mu(D)} \int_D f \,\mathrm{d}\mu - x\right| &= \frac{1}{\mu(D)} \left|\int_D (f - x) \,\mathrm{d}\mu\right| \\ &\leq \frac{1}{\mu(D)} \int_D |f - x| \,\mathrm{d}\mu < \frac{1}{\mu(D)} \int_D r \,\mathrm{d}\mu = r.\end{aligned}$$

In particular, $\frac{1}{\mu(D)} \int_D f \,\mathrm{d}\mu \in B_n$. But $B_n \cap S = \emptyset$, so we get a contradiction with the assumptions. □

Exercise 25. 1. Explain where in the above proof σ-finiteness was used. 2. Extend the proof to the case where μ is not σ-finite by replacing $f(x) \in S$ with $f(x) \in S \cup \{0\}$ in the statement of the Theorem.

Lemma 5.39. *Let $f \in \mathcal{L}^1$ and assume $\int_A f = 0$ for all measureable sets A. Then, $f = 0$ almost everywhere.*

Proof. **Exercise.** □

Lemma 5.40. *Let X be a measure space with σ-finite measure μ and let $p \in (0, \infty)$. Then, there exists a function $w \in \mathcal{L}^p(X, \mu)$ such that $0 < w < 1$.*

Proof. Let $\{X_n\}_{n \in \mathbb{N}}$ be a sequence of disjoint sets of finite measure such that $X = \bigcup_{n \in \mathbb{N}} X_n$. Define

$$w(x) := \left(\frac{2^{-n}}{1 + \mu(X_n)}\right)^{1/p} \quad \text{if } x \in X_n.$$

This has the desired properties. **Exercise.**Show this. □

Theorem 5.41. *Let X be a measure space with σ-algebra $\mathcal{M}$ and σ-finite measure μ. Let ν be a finite measure on $(X, \mathcal{M})$.*

1. *(Lebesgue) Then, there exists a unique decomposition*
$$\nu = \nu_a + \nu_s,$$
into finite measures such that $\nu_a \ll \mu$ and $\nu_s \perp \mu$.

2. *(Radon-Nikodym) There exists a unique $[h] \in \mathrm{L}^1(\mu)$ such that for all $A \in \mathcal{M}$,*
$$\nu_a(A) = \int_A h \, \mathrm{d}\mu.$$

Proof. We first show the uniqueness of the decomposition $\nu = \nu_a + \nu_s$ in (1.). Suppose there is another decomposition $\nu = \nu'_a + \nu'_s$. Note that all the measures involved here are finite and thus are also complex measures. In particular, we obtain the following equality of complex measures, $\nu_a - \nu'_a = \nu'_s - \nu_s$. However, by Proposition 5.37 the left hand side is absolutely continuous with respect to μ while the right hand side is singular with respect to μ. Again by Proposition 5.37, the equality of both sides implies that they must be zero, i.e., $\nu'_a = \nu_a$ and $\nu'_s = \nu_s$.

To show the uniqueness of $[h] \in \mathrm{L}^1(\mu)$ in (2.) we note that given another element $[h'] \in \mathrm{L}^1(\mu)$ with the same property, we would get $\int_A (h - h') \, \mathrm{d}\mu = 0$ for all measurable sets A. By Lemma 5.39 then $0 = [h - h'] = [h] - [h'] \in \mathrm{L}^1(\mu)$.

We proceed to construct the decomposition $\nu = \nu_a + \nu_s$ and the element $[h] \in \mathrm{L}^1(\mu)$. By Lemma 5.40, there is a function $w \in \mathcal{L}^1(\mu)$ with $0 < w < 1$. This yields the finite measure μ_w, given by
$$\mu_w(A) := \int_A w \, \mathrm{d}\mu \quad \forall A \in \mathcal{M}.$$
(Recall the last part of Exercise 22.) Define the finite measure $\varphi := \nu + \mu_w$. Note that $\mathcal{L}^1(\varphi) \subseteq \mathcal{L}^1(\nu)$ and $\mathcal{L}^1(\varphi) \subseteq \mathcal{L}^1(\mu_w)$ and we have (using Proposition 5.31),
$$\int_X f \, \mathrm{d}\varphi = \int_X f \, \mathrm{d}\nu + \int_X f w \, \mathrm{d}\mu \quad \forall f \in \mathcal{L}^1(\varphi). \tag{1}$$
In particular, we may deduce
$$\left| \int_X f \mathrm{d}\nu \right| \le \|f\|_{\nu,1} \le \|f\|_{\varphi,1} \quad \forall f \in \mathcal{L}^1(\varphi).$$
By Proposition 5.23 [and its extension seen in class] we have $\mathcal{L}^2(\varphi) \subseteq \mathcal{L}^1(\varphi)$ and even
$$\|f\|_{\varphi,1} \le \|f\|_{\varphi,2} \, (\varphi(X))^{1/2} \quad \forall f \in \mathcal{L}^2(\varphi).$$

Combining the inequalities we find

$$\left|\int_X f\,\mathrm{d}\nu\right| \le \|f\|_{\varphi,2}\,(\varphi(X))^{1/2} \quad \forall f \in \mathcal{L}^2(\varphi).$$

This means that the linear map $\alpha : \mathrm{L}^2(\varphi) \to \mathbb{K} \subseteq \mathbb{C}$ given by $[f] \mapsto \int_X [f]\mathrm{d}\nu$ is bounded. Since $\mathrm{L}^2(\varphi)$ is a Hilbert space, Theorem 5.29 implies that there is an element $g \in \mathcal{L}^2(\varphi)$ such that $\alpha([f]) = \langle [f], [\overline{g}]\rangle$ for all $f \in \mathcal{L}^2(\varphi)$. This implies,

$$\int_X f\,\mathrm{d}\nu = \int_X f g\,\mathrm{d}\varphi \quad \forall f \in \mathcal{L}^2(\varphi) \tag{2}$$

By inserting characteristic functions for f we obtain

$$\nu(A) = \int_A g\,\mathrm{d}\varphi \quad \forall A \in \mathcal{M}.$$

On the other hand we have $\nu(A) \le \varphi(A)$ for all measurable sets A and hence,

$$0 \le \frac{1}{\varphi(A)} \int_A g\,\mathrm{d}\varphi = \frac{\nu(A)}{\varphi(A)} \le 1 \quad \forall A \in \mathcal{M} : \varphi(A) > 0.$$

We can now apply the Averaging Theorem (Theorem 5.38) to conclude that $0 \le g \le 1$ almost everywhere. We modify g on a set of measure zero if necessary so that $0 \le g \le 1$ everywhere. In particular, if $f \in \mathcal{L}^2(\varphi)$ then $(1-g)f \in \mathcal{L}^2(\varphi)$ and $gf \in \mathcal{L}^2(\varphi)$. Combining (1) and (2) we find

$$\int_X (1-g) f\,\mathrm{d}\nu = \int_X f g w\,\mathrm{d}\mu \quad \forall f \in \mathcal{L}^2(\varphi).$$

Set $Z_a := \{x \in X : g(x) < 1\}$ and $Z_s := \{x \in X : g(x) = 1\}$ and define the measures $\nu_a(A) := \nu(A \cap Z_a)$ and $\nu_s := \nu(A \cap Z_s)$ for all $A \in \mathcal{M}$. Since X is the disjoint union of Z_a and Z_s we obviously have $\nu = \nu_a + \nu_s$. Taking f to be the characteristic function of Z_s we find that $\int_{Z_s} w\,\mathrm{d}\mu = 0$. Since $0 < w$, we conclude that $\mu(Z_s) = 0$. In particular, this implies that μ is supported on Z_a, while ν_s is supported on Z_s, so $\nu_s \perp \mu$.

Define now the sequence $\{f_n\}_{n\in\mathbb{N}}$ of functions $f_n := \sum_{k=1}^n g^{k-1}$. Since g is bounded, f_n is bounded. Multiplying with characteristic functions we find for measurable sets A,

$$\int_A (1-g^n)\,\mathrm{d}\nu = \int_A (1-g) f_n\,\mathrm{d}\nu = \int_A f_n g w\,\mathrm{d}\mu.$$

Note that $\{1-g^n\}_{n\in\mathbb{N}}$ increases monotonically and converges pointwise to the characteristic function of Z_a. Thus, by the Monotone Convergence Theorem (Theorem 4.23) or by the Dominated Convergence Theorem (Theorem 4.26) the left hand side converges to $\nu(A \cap Z_a) = \nu_a(A)$.

The sequence $\{f_n g w\}_{n\in\mathbb{N}}$ is also increasing monotonically with its μ-integrals over A bounded by $\nu_a(A)$. So the Monotone Convergence Theorem

(Theorem 4.23) applies and the pointwise limit is a μ-integrable function h. We get

$$\nu_a(A) = \int_A h \, \mathrm{d}\mu,$$

showing existence in (2.) and also $\nu_a \ll \mu$, thus completing the existence proof for (1.). $\square$

Remark 5.42. The function h appearing in the above Theorem is also called the *Radon-Nikodym derivative*, denoted as $h = \mathrm{d}\nu_a/\mathrm{d}\mu$.

Exercise 26 (adapted from Lang)**.** Let X be a measure space with σ-finite measure μ and let $p \in [1, \infty)$. Let $T : \mathrm{L}^p \to \mathrm{L}^p$ be a bounded linear map. For each $g \in \mathrm{L}^\infty$ consider the bounded linear map $M_g : \mathrm{L}^p \to \mathrm{L}^p$ given by $f \mapsto gf$. Assume that T and M_g commute for all $g \in \mathrm{L}^\infty$, i.e., $T \circ M_g = M_g \circ T$. Show that $T = M_h$ for some $h \in \mathrm{L}^\infty$. [Hint: Use Lemma 5.40 to obtain a function $w \in \mathrm{L}^p \cap \mathrm{L}^\infty$ with $0 < w$. Then, for $f \in \mathrm{L}^p \cap \mathrm{L}^\infty$ we have

$$T(wf) = wT(f) = fT(w).$$

If we define $h := T(w)/w$ we thus have $T(f) = hf$. Prove that h is essentially bounded by contradiction: Assume it is not and consider sets of positive measure where $|h| > c$ for some constant c and evaluate T on the characteristic function of such sets. Finally, prove that $T(f) = hf$ for all $f \in \mathrm{L}^p$.]